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J. Differential Equations 189 (2003) 234–266

Journal of Differential Equations

http://www.elsevier.com/locate/jde

Restrictions and unfolding of double Hopf bifurcation in functional differential equations

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Received March 18, 2002; revised September 18, 2002

Abstract

The normal form of a vector field generated by scalar delay-differential equations at nonresonant double Hopf bifurcation points is investigated. Using the methods developed by Faria and Magalhães (J. Differential Equations 122 (1995) 181) we show that (1) there exists linearly independent unfolding parameters of classes of delay-differential equations for a double Hopf point which generically map to linearly independent unfolding parameters of the normal form equations (ordinary differential equations), (2) there are generically no restrictions on the possible flows near a double Hopf point for both general and $\mathbf{Z}_2$-symmetric first-order scalar equations with two delays in the nonlinearity, and (3) there always are restrictions on the possible flows near a double Hopf point for first-order scalar delay-differential equations with one delay in the nonlinearity, and in nth-order scalar delay-differential equations ($n \geqslant 2$) with one delay feedback.

Keywords: Delay-differential equations; Double Hopf bifurcation; Normal forms; Centre manifold; Unfolding

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0022-0396/03/$ - see front matter
doi:10.1016/S0022-0396(02)00179-1

1. Introduction

Delay-differential equations share many (but not all) properties with ordinary differential equations. This analogy has been made more precise and put on solid theoretical ground as the methods and techniques of geometric dynamical systems theory have been implemented in functional differential equations, see [14] for numerous references and comments. In particular, invariant manifolds for the flow associated with an equation near an equilibrium point have been established, along with their uniqueness and smoothness properties of the manifolds. At a bifurcation point, the flow near the equilibrium of the delay-differential equation is essentially governed by the vector field on the centre manifold. In this paper, we investigate the flow near double Hopf bifurcation points in scalar first-order and nth-order scalar delay-differential equations by studying the flow on the centre manifold using normal form theory.

The redeeming feature of centre manifold calculations is the possibility of unfolding degenerate flows in the neighbourhood of invariant sets, in general, and of stationary points in particular. In unfolding the flow on the centre manifold, a number of theoretical questions arise. The unfolding itself takes place in the framework of ordinary differential equations, for which most lower codimension cases have been solved [12]. For a given class of delay-differential equations, it is not a priori obvious, given the circumvoluted reduction procedure involved, that the unfolding of the reduced flow can be obtained from an unfolding of the class of delay-differential equations. Faria and Magalhães [9] determine parameter families of scalar first-order equations leading to reduced flows with appropriate unfolding parameters for several singularities: Hopf, Bogdanov–Takens and steady-state/Hopf. We find such parameter families of scalar first-order and nth-order delay equations for the double Hopf bifurcation, see Theorems 3.1 and 4.1.

A natural question concerns the possible restrictions on the flows that can occur on the centre manifold after reduction. In this paper, we study this question at double Hopf bifurcation points for the above mentioned classes of delay-differential equations. This question has been answered in part by Faria and Magalhães [8]. They show that any finite jet of an ordinary differential equation can be realized as the centre manifold reduction from a delay-differential equation in $\mathbf{R}^n$ where n is large enough and the nonlinearity depends on sufficiently many delays. Realizability can still be achieved when the number of delays is not sufficient, and this situation is studied by Faria and Magalhães [9] for scalar first-order delay-differential equations near Hopf, Bogdanov–Takens and steady-state/Hopf bifurcation points. In particular, realizability holds for the Hopf and Bogdanov–Takens points with one delay and generically for the steady-state/Hopf with two delays. However, there are strong restrictions on the possible flows near a steady-state/Hopf bifurcation point if the nonlinearity depends on a single delay. Recently, Redmond et al. [16] study the Bogdanov–Takens bifurcation with reflectional symmetry in a scalar first-order delay equation with one delay and show that there are no restrictions on the possible phase portraits.

The determination of possible unfoldings is quite different in a modelling context since it may be leading to different conditions, as pointed out in Hale [13]. This becomes particularly significant if our interest lies not so much in assessing *all* possible behaviours in a class of systems, but rather in trying to determine the range of dynamics accessible in a specific model which depends on a number of parameters. The form of the model then becomes a crucial factor in this determination of possible invariant sets, for example.

For double Hopf bifurcation points, by Theorem 4.5 of [8], the vector field on the centre manifold can be realized by a scalar first-order delay-differential equation where the number of delays is 4. We study double Hopf bifurcations in scalar first-order delay-differential equations with one and two delays and in nth-order scalar differential equation with delayed feedback. We show that, generically, there are no restrictions on the possible flows near a double Hopf bifurcation point for $\mathbf{Z}_2$-symmetric and general scalar first-order delay-differential equations depending on two delays in the nonlinearity. If only one delay is present in the nonlinearity, we compute the normal form to cubic order and show that there always are restrictions on the possible phase portraits, see Theorem 3.1.

We study in more detail the equation considered by Bélair and Campbell [2]: they identify, in the $\mathbf{Z}_2$-symmetric scalar equation

$$\dot{x}(t) = -A_1 \tanh(x(t-\tau_1)) - A_2 \tanh(x(t-\tau_2)) \tag{1}$$

points of double Hopf bifurcation at the boundary of the region of linear stability in the space of the parameters $(A_1, A_2, \tau_1, \tau_2)$. Using centre manifold calculations, they find restrictions on the possible phase portraits that can appear in the neighbourhood of this singularity. We show that these restrictions are due in part to the $\mathbf{Z}_2$ symmetry and to the particular form of (1). We consider equations exhibiting that symmetry in details, recovering and generalizing results from [2].

Finally, we study the normal form of the double Hopf bifurcation in nth-order scalar delay-differential equations. A particular example of such equations is the harmonic oscillator with delayed feedback studied by Campbell et al. [4]. We show that the cubic normal form on the centre manifold is given by expressions similar to the cubic normal form for the scalar first-order equation with one delay in the nonlinearity. Therefore, the same restrictions as for first-order equations with one delay apply in this case, see Theorem 4.1.

The explicit flow induced by a class of specific functional differential equations on a centre manifold has been made accessible by recent advances in computing power: these calculations have been implemented using symbolic (or analytic) computations, first with Macsyma [10] and more recently with Maple [3]. In the computation of normal forms of a reduced flow on a centre manifold, Bélair and Campbell [2] used an approach in two steps: they first computed the centre manifold, and then projected the flow from the delay equation on the manifold, then computing the corresponding normal form. Faria and Magalhães [6], however, used a different approach, which is the one we employ here: they compute in a single procedure both the centre manifold and the normal form of the flow projected on it.

Our analysis is the first complete investigation of the double Hopf bifurcation as it occurs in delay-differential equations, and the relationship between unfolded flows on a four-dimensional centre manifold and the original delay-differential equation: all previous analysis of the restriction question [6,8,9,16] only address unfolding on centre manifolds of dimension three or less.

The paper is organized as follows. Our main results are summarized in Theorems 3.1 and 4.1. The next section is a review of normal form theory for functional differential equations and in particular of the double Hopf bifurcation. The proof of Theorem 3.1 is given in Section 3 and the proof of Theorem 4.1 is given in Section 4. We conclude with a summary and a discussion of our results. Some more tedious normal form computations are relegated to the appendix.

2. Normal form for delay-differential equations

We first recall standard results to fix notation, see [14]. Let $C = C([-r,0],\mathbf{R}^n)$, $L: C \times \mathbf{R}^p \to \mathbf{R}^n$ be a continuous linear map and $F:\ \mathbf{R}^n \times \mathbf{R}^p \to \mathbf{R}^n$ a smooth map. Consider the retarded functional differential equation:

$$\dot{z}(t) = L(\mu)z_t + F(z_t,\mu), \tag{2}$$

where $z_t \in C$ is defined as $z_t(\theta) = z(t+\theta)$ with $\theta \in [-r,0]$. The linear map $L(\mu)$ may be expressed in integral form as

$$L(\mu)\phi = \int_{-r}^{0} [d\eta(\theta)]\phi(\theta),$$

where $\eta : [-r,0] \to \mathbf{R}^n$ is a function of bounded variation. Let $L_0 = L(0)$, and rewrite (2) to exhibit the parameters in the linear map:

$$\dot{z}(t) = L_0 z_t + [L(\mu) - L_0]z_t + F(z_t,\mu). \tag{3}$$

Let $A(\mu)$ be the infinitesimal generator for the flow of the linear system

$$\dot{z} = L(\mu)z_t.$$

Let $\sigma(A(\mu))$ denote the spectrum of $A(\mu)$ and Λ_μ be the set of eigenvalues of $\sigma(A(\mu))$ with zero real part. The bilinear form

$$(\psi,\phi) = \psi(0)\phi(0) - \int_{-r}^{0}\int_{0}^{\theta} \psi(\xi-\theta)\, d\eta(\theta)\phi(\xi)\, d\xi \tag{4}$$

is used to decompose C as $C = P \oplus Q$ where P is the generalized eigenspace of the eigenvalues in Λ_0 and Q is an infinite dimensional complementary subspace. A basis for P is given by $\Phi_{\Lambda_0} = \{\Phi_{\lambda_1},\ldots,\Phi_{\lambda_m}\}$ and denote by B be the finite dimensional matrix of the restriction of A to $\Phi_{\Lambda_0} : A\Phi_{\Lambda_0} = \Phi_{\Lambda_0}B$. The set $\Psi = \mathrm{col}\{\Psi_1,\ldots,\Psi_m\}$ is a basis of the dual space P^* in C^* with $(\Psi,\Phi) = I$, the identity matrix.

Faria and Magalhães [7] show that Eq. (3) can be written as an ordinary differential equation on the Banach space BC of functions from $[-r, 0]$ to $\mathbf{R}^n$ bounded and continuous on $[-r, 0)$ with a possible jump discontinuity at 0. Elements of BC are of the form $\phi + X_0\alpha$ where $\phi \in C$, $\alpha \in \mathbf{R}^n$ and $X_0(\theta) = 0$ for $\theta \in [-r, 0)$ and $X_0(0) = I$. Let $\pi : BC \to P$ be a continuous projection defined by $\pi(\phi + X_0\alpha) = \Phi[(\Psi, \phi) + \Psi(0)\alpha]$. We can write $BC = P \oplus \ker \pi$ with the property that $Q \subsetneq \ker \pi$. Decompose $z_t = \Phi x_t + y_t$ where $x_t \in \mathbf{R}^m$ and $y_t \in \ker \pi \cap D(A) \equiv Q^1$ where $D(A)$ is the domain of A. Eq. (3) is equivalent to system

$$\dot{x} = Bx + \Psi(0)\{[L(\mu) - L_0](\Phi x + y) + F(\Phi x + y, \mu)\},$$

$$\dot{y} = A_{Q^1} y + (I - \pi)X_0\{[L(\mu) - L_0](\Phi x + y) + F(\Phi x + y, \mu)\}, \tag{5}$$

where $A_{Q^1} : Q^1 \to \ker \pi$ is such that $A_{Q^1}\phi = \dot{\phi} + X_0[L(\phi) - \dot{\phi}(0)]$. Let F_j be the jth Fréchet derivative of F, we take the Taylor expansion of F which transforms (5) to

$$\dot{x} = Bx + \sum_{j \geqslant 2} \frac{1}{j!} f_j^1(x, y, \mu),$$

$$\dot{y} = A_{Q^1} y + \sum_{j \geqslant 2} \frac{1}{j!} f_j^2(x, y, \mu), \tag{6}$$

where $f_j^1(x, y, \mu)$ and $f_j^2(x, y, \mu)$ are the homogeneous polynomials of degree j in the Taylor expansions of the nonlinear terms of $\dot{x}$ and $\dot{y}$ in (5).

Eq. (2) is said to satisfy *nonresonance conditions relative to* Λ_μ if $(q, \tilde{\lambda}) \neq \eta$ for all $\eta \in \sigma(A_0) \backslash \Lambda_\mu$, where q is an m-tuple of nonnegative integers, $|q| \geqslant 2$ and $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_m)$. For the remainder of the paper, we assume the following hypotheses:

H1. $\mathrm{Card}(\Lambda_\mu) < \mathrm{Card}(\Lambda_0)$ for μ small.
H2. Eq. (2) satisfies the nonresonance conditions relative to Λ_0.

Under hypotheses H1 and H2, Faria and Magalhães show that system (6) can be put in formal normal form

$$\dot{x} = Bx + \sum_{j \geqslant 2} \frac{1}{j!} g_j^1(x, y, \mu),$$

$$\dot{y} = A_{Q^1} y + \sum_{j \geqslant 2} \frac{1}{j!} g_j^2(x, y, \mu), \tag{7}$$

such that the centre manifold is locally given by $y = 0$ and the equation for the vector field on the centre manifold is

$$\dot{x} = Bx + \sum_{j \geqslant 2} \frac{1}{j!} g_j^1(x, 0, \mu).$$

2.1. Double Hopf bifurcation

A nonresonant double Hopf bifurcation occurs if the linearization L_0 has a pair of eigenvalues $\pm i\omega_1$, $\pm i\omega_2$ with $\omega_1/\omega_2 \notin \mathbf{Q}$. We can assume that all other eigenvalues have negative real parts. This assumption is reasonable since in the cases of interest in this paper, Bélair and Campbell [2] and Campbell et al. [4] show that points of double Hopf bifurcation lie at the boundary of the stability region. The critical set of eigenvalues is $\Lambda = \{i\omega_1, -i\omega_1, i\omega_2, -i\omega_2\}$ with eigenspace P. The restriction of L_0 to P is the matrix B defined above. In complex coordinates B is diagonal:

$$B = \begin{bmatrix} i\omega_1 & 0 & 0 & 0 \\ 0 & -i\omega_1 & 0 & 0 \\ 0 & 0 & i\omega_2 & 0 \\ 0 & 0 & 0 & -i\omega_2 \end{bmatrix}, \tag{8}$$

which simplifies the normal form transformations.

The matrix B generates the torus group $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$ whose action on $\mathbf{C}^2$ is given by

$$(\theta_1, \theta_2)(z_1, z_2) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2).$$

Elphick et al. [5] show a possible normal form that commutes with the action of $\mathbf{T}^2$ described above. We use this normal form for the double Hopf bifurcation without symmetry and with $\mathbf{Z}_2$ symmetry. The formal normal form is the following, see [11]:

$$\begin{aligned} \dot{z}_1 &= p_1(|z_1|^2, |z_2|^2) z_1, \\ \dot{z}_2 &= p_2(|z_1|^2, |z_2|^2) z_2. \end{aligned}$$

Truncating the normal form equation to degree three we obtain

$$\begin{aligned} \dot{z}_1 &= (i\omega_1 + c_{11}|z_1|^2 + c_{12}|z_2|^2) z_1, \\ \dot{z}_2 &= (i\omega_2 + c_{21}|z_1|^2 + c_{22}|z_2|^2) z_2, \end{aligned} \tag{9}$$

where c_{11}, c_{22}, c_{12}, c_{21} are complex numbers. Takens [17] shows that nonresonant double Hopf bifurcation is determined to third order if the nondegeneracy conditions $\mathrm{Re}(c_{ij}) \neq 0$, $i = 1, 2$ and $\mathrm{Re}(c_{11})\mathrm{Re}(c_{22}) - \mathrm{Re}(c_{12})\mathrm{Re}(c_{21}) \neq 0$ are satisfied.

Let $z_1 = r_1 e^{i\rho_1}$ and $z_2 = r_2 e^{i\rho_2}$. The phase/amplitude equations corresponding to (9) are

$$\begin{aligned}
\dot{r}_1 &= (\mathrm{Re}(c_{11})r_1^2 + \mathrm{Re}(c_{12})r_2^2)r_1,\\
\dot{r}_2 &= (\mathrm{Re}(c_{21})r_1^2 + \mathrm{Re}(c_{22})r_2^2)r_2,\\
\dot{\rho}_1 &= \omega_1 + \mathrm{Im}(c_{11})r_1^2 + \mathrm{Im}(c_{12})r_2^2,\\
\dot{\rho}_2 &= \omega_2 + \mathrm{Im}(c_{21})r_1^2 + \mathrm{Im}(c_{22})r_2^2.
\end{aligned} \tag{10}$$

The possible phase portraits in a neighbourhood of a double Hopf point are classified by the dynamics of the planar system given by the amplitude equations $(\dot{r}_1, \dot{r}_2)$.

Let the system depend on parameters (η_1, η_2). Then the $\mathbf{T}^2$ action on $\mathbf{R}^2 \times \mathbf{C}^2$ is given by

$$(\theta_1, \theta_2)(\eta_1, \eta_2, z_1, z_2) = (\eta_1, \eta_2, e^{i\theta_1} z_1, e^{i\theta_2} z_2).$$

Then the $\mathbf{T}^2$-equivariant normal form with parameters is

$$\begin{aligned}
\dot{z}_1 &= p_1(\eta_1, \eta_2, |z_1|^2, |z_2|^2)z_1,\\
\dot{z}_2 &= p_2(\eta_1, \eta_2, |z_1|^2, |z_2|^2)z_2.
\end{aligned} \tag{11}$$

The truncation to quadratic order is

$$\begin{aligned}
\dot{z}_1 &= i\omega_1 z_1 + \alpha_1\eta_1 z_1 + \alpha_2\eta_2 z_1,\\
\dot{z}_2 &= i\omega_2 z_2 + \beta_1\eta_1 z_2 + \beta_2\eta_2 z_2.
\end{aligned} \tag{12}$$

Letting $\mu_1 = \alpha_1\eta_1 + \alpha_2\eta_2$ and $\mu_2 = \beta_1\eta_1 + \beta_2\eta_2$, the amplitude equations to cubic order are

$$\begin{aligned}
\dot{r}_1 &= (\mu_1 + \mathrm{Re}(c_{11})r_1^2 + \mathrm{Re}(c_{12})r_2^2)r_1,\\
\dot{r}_2 &= (\mu_2 + \mathrm{Re}(c_{21})r_1^2 + \mathrm{Re}(c_{22})r_2^2)r_2,
\end{aligned} \tag{13}$$

where μ_1 and μ_2 are unfolding parameters (generically independent).

Since B is diagonal, the monomials of the normal form (11) are the resonant monomials. The following result, see [1], guarantees that the coefficients of the resonant monomials before and after normal form transformation are equal. The proof is a simple computation.

Proposition 2.1. *Consider the system of differential equations in* $\mathbf{C}^n$,

$$\dot{x} = Ax + f^2(x) + \cdots + f^r(x)$$

with $A = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ *and*

$$f^k(x) = \sum_{i=1}^{n} \sum_{|\alpha|=k} C_{\alpha i} x^\alpha e_i,$$

where $\{e_1, \dots, e_n\}$ *is the standard basis of* $\mathbf{C}^n$. *Then the nonlinear normal form transformation* $x = y + \sigma^k(y)$ *where*

$$\sigma^k(y) = \sum_{i=1}^{n} \left(\sum_{\lambda \cdot \alpha \neq \lambda_i} \frac{C_{\alpha i}}{\lambda \cdot \alpha - \lambda_i} y^\alpha e_i + \sum_{\lambda \cdot \alpha = \lambda_i} p_{\alpha i} y^\alpha e_i \right)$$

and $\lambda \cdot \alpha = \sum_{i=1}^{n} \lambda_i \alpha_i$ *transforms the terms of degree* k *into*

$$g^k(y) = \sum_{i=1}^{n} \sum_{\lambda \cdot \alpha = \lambda_i} C_{\alpha i} y^\alpha e_i.$$

3. First-order scalar delay-differential equations

We study the restriction on the normal form at a nonresonant double Hopf bifurcation point for the following delay-differential equations:

$$\dot{u} = L(u_t) + f(u(t-\tau_1), u(t-\tau_2)), \tag{14}$$

$$\dot{u} = L(u_t) + f_1(u(t-\tau_1)^2, u(t-\tau_2)^2)u(t-\tau_1)$$

$$+ f_2(u(t-\tau_1)^2, u(t-\tau_2)^2)u(t-\tau_2), \tag{15}$$

$$\dot{u} = L(u_t) + f(u(t-\tau)). \tag{16}$$

For each equation, $L(u_t) = a_{10}u(t-\tau_1) + a_{01}u(t-\tau_2)$, and for (14) and (15), $f(0,0) = Df(0,0) = 0$ while for (16), $f(0) = Df(0) = 0$. Eq. (14) is a general equation depending on two delays. Eq. (15) is a $\mathbf{Z}_2$-symmetric equation depending also on two delays, it is a generalization of the system studied by Bélair and Campbell [2]. Eq. (16) has a nonlinearity depending on only one delay. The following result is proved in this section.

Theorem 3.1. *Suppose that Eqs.* (14), (15), *or* (16) *has a nonresonant double Hopf bifurcation point at the origin. Then, generically, the two parameter family*

$$\dot{u} = (a_{10} + v_1)u(t-\tau_1) + (a_{01} + v_2)u(t-\tau_2) + o(u(t-\tau_1), u(t-\tau_2)) \tag{17}$$

is a universal unfolding for the double Hopf bifurcation. Moreover,

(1) *for* (14) *and* (15), *generically, there are no restrictions on the possible phase portraits near the double Hopf point, and*

(2) *for* (16) *there always are restrictions on the possible phase portraits near the double Hopf bifurcation.*

The proof of the unfolding part is given in Proposition 3.2. The proof of (1) is given in Propositions 3.3 and 3.5. Finally, the proof of (2) is given in Proposition 3.9.

3.1. The $C = P \oplus Q$ decomposition

In this section, we write systems (14)–(16) as infinite dimensional systems. The bases of P and P^* are, respectively,

$$\begin{aligned}\Phi(\theta) &= (e^{i\omega_1\theta}, e^{-i\omega_1\theta}, e^{i\omega_2\theta}, e^{-i\omega_2\theta}),\\ \Psi(s) &= (\psi_1(0)e^{-i\omega_1 s}, \psi_2(0)e^{i\omega_1 s}, \psi_3(0)e^{-i\omega_2 s}, \psi_4(0)e^{i\omega_2 s})^t,\end{aligned}$$

where

$$\begin{aligned}\psi_1(0) &= [1 - L(\theta e^{i\omega_1\theta})]^{-1}\psi_2(0) = \overline{\psi_1(0)},\\ \psi_3(0) &= [1 - L(\theta e^{i\omega_2\theta})]^{-1}\psi_4(0) = \overline{\psi_3(0)}.\end{aligned}$$

Note that $\psi_1(0)$ and $\psi_3(0)$ are identical functions of ω_1 and ω_2, respectively.

Truncate (14)–(16) to cubic order. Let F_2 and F_3 be homogeneous polynomials of degree two and three, respectively. Then the two-delay equations are

$$\dot{u} = L(v_1, v_2)u_t + F_2(u(t-\tau_1), u(t-\tau_2)) + F_3(u(t-\tau_1), u(t-\tau_2)),$$

where for Eq. (15), $F_2 \equiv 0$. The one-delay equation is

$$\dot{u} = L(v_1, v_2)u_t + F_2(u(t-\tau)) + F_3(u(t-\tau)).$$

Let $z = (z_1, \bar{z}_1, z_2, \bar{z}_2)^t$ and $y \in Q^1 = Q \cap C^1([-h, 0], \mathbf{R})$ where $h = \max(\tau, \tau_1, \tau_2)$, then system (6), up to degree three, for the three first-order equations is

$$\begin{cases} \dot{z}_1 = i\omega_1 z_1 + \psi_1(0)([L(v_1, v_2) - L_0](\Phi z + y) + F_2(\Phi z + y) + F_3(\Phi z + y)),\\ \dot{\bar{z}}_1 = -i\omega_1 \bar{z}_1 + \psi_2(0)\overline{([L(v_1, v_2) - L_0](\Phi z + y) + F_2(\Phi z + y) + F_3(\Phi z + y))},\\ \dot{z}_2 = i\omega_2 z_2 + \psi_3(0)([L(v_1, v_2) - L_0](\Phi z + y) + F_2(\Phi z + y) + F_3(\Phi z + y)),\\ \dot{\bar{z}}_2 = -i\omega_2 \bar{z}_2 + \psi_4(0)\overline{([L(v_1, v_2) - L_0](\Phi z + y) + F_2(\Phi z + y) + F_3(\Phi z + y))},\\ \dfrac{dy}{dt} = A_{Q^1}y + (I - \pi)X_0([L(v_1, v_2) - L_0](\Phi z + y) + F_2(\Phi z + y) + F_3(\Phi z + y)). \end{cases} \tag{18}$$

If we remove the dependence on the unfolding parameters, we obtain

$$\dot{z}_1 = i\omega_1 z_1 + \psi_1(0)(F_2(\Phi z + y) + F_3(\Phi z + y)),$$

$$\dot{\bar{z}}_1 = -i\omega_1 \bar{z}_1 + \psi_2(0)\overline{F_2(\Phi z + y) + F_3(\Phi z + y))},$$

$$\dot{z}_2 = i\omega_2 z_2 + \psi_3(0)(F_2(\Phi z + y) + F_3(\Phi z + y)),$$

$$\dot{\bar{z}}_2 = -i\omega_2 \bar{z}_2 + \psi_4(0)\overline{F_2(\Phi z + y) + F_3(\Phi z + y))},$$

$$\frac{dy}{dt} = A_{Q^1} y + (I - \pi)X_0(F_2(\Phi z + y) + F_3(\Phi z + y)). \tag{19}$$

3.2. Unfolding of the first-order equations

The linear equation with unfolding parameters is

$$L(v_1, v_2)u_t = (a_{10} + v_1)u(t - \tau_1) + (a_{01} + v_2)u(t - \tau_2). \tag{20}$$

Let $L_0 = L(0,0)$. The quadratic truncation of (18) in the z_1 and z_2 variables at $y = 0$ is

$$\dot{z}_1 = i\omega_1 z_1 + \psi_1(0)(v_1\Phi(-\tau_1)z + v_2\Phi(-\tau_2)z + F_2(\Phi(-\tau_1)z, \Phi(-\tau_2)z)),$$

$$\dot{z}_2 = i\omega_1 z_2 + \psi_3(0)(v_1\Phi(-\tau_1)z + v_2\Phi(-\tau_2)z + F_2(\Phi(-\tau_1)z, \Phi(-\tau_2)z)). \tag{21}$$

By Eq. (12) and Proposition 2.1, the normal form to quadratic order is given by

$$\dot{z}_1 = i\omega_1 z_1 + \psi_1(0)(e^{-i\omega_1\tau_1}v_1 + e^{-i\omega_1\tau_2}v_2)z_1,$$

$$\dot{z}_2 = i\omega_2 z_2 + \psi_3(0)(e^{-i\omega_2\tau_1}v_1 + e^{-i\omega_2\tau_2}v_2)z_2, \tag{22}$$

where the coefficients of the resonant monomials are unchanged by the normal form transformation from (21). In polar coordinates $z_1 = r_1 e^{\rho_1}$ and $z_2 = r_2 e^{\rho_2}$, the amplitude equation coming from (22) is

$$\begin{aligned}\dot{r}_1 &= (\mathrm{Re}(\psi_1(0)e^{-i\omega_1\tau_1})v_1 + \mathrm{Re}(\psi_1(0)e^{-i\omega_1\tau_2})v_2)r_1,\\ \dot{r}_2 &= (\mathrm{Re}(\psi_3(0)e^{-i\omega_2\tau_1})v_1 + \mathrm{Re}(\psi_3(0)e^{-i\omega_2\tau_2})v_2)r_2.\end{aligned}$$

Let

$$\begin{aligned}\mu_1 &= \mathrm{Re}(\psi_1(0)e^{-i\omega_1\tau_1})v_1 + \mathrm{Re}(\psi_1(0)e^{-i\omega_1\tau_2})v_2,\\ \mu_2 &= \mathrm{Re}(\psi_3(0)e^{-i\omega_2\tau_1})v_1 + \mathrm{Re}(\psi_3(0)e^{-i\omega_2\tau_2})v_2.\end{aligned}$$

Proposition 3.2. *Generically, the independent unfolding parameters* (v_1, v_2) *of* (17) *map to independent unfolding parameters* (μ_1, μ_2) *of the normal form equations.*

Proof. Let

$$Q = \begin{bmatrix} \mathrm{Re}(\psi_1(0)e^{-i\omega_1\tau_1}) & \mathrm{Re}(\psi_1(0)e^{-i\omega_1\tau_2}) \\ \mathrm{Re}(\psi_3(0)e^{-i\omega_2\tau_1}) & \mathrm{Re}(\psi_3(0)e^{-i\omega_2\tau_2}) \end{bmatrix}.$$

Then

$$\begin{aligned} \det Q = {} & \mathrm{Re}(\psi_1(0))\mathrm{Re}(\psi_3(0))(\cos(\omega_1\tau_1)\cos(\omega_2\tau_2) - \cos(\omega_1\tau_2)\cos(\omega_2\tau_1)) \\ & + \mathrm{Re}(\psi_1(0))\,\mathrm{Im}(\psi_3(0))(\cos(\omega_1\tau_1)\sin(\omega_2\tau_2) - \cos(\omega_1\tau_2)\sin(\omega_2\tau_1)) \\ & + \mathrm{Im}(\psi_1(0))\,\mathrm{Re}(\psi_3(0))(\sin(\omega_1\tau_1)\cos(\omega_2\tau_2) - \sin(\omega_1\tau_2)\cos(\omega_2\tau_1)) \\ & + \mathrm{Im}(\psi_1(0))\,\mathrm{Im}(\psi_3(0))(\sin(\omega_1\tau_1)\sin(\omega_2\tau_2) - \sin(\omega_1\tau_2)\sin(\omega_2\tau_1)). \end{aligned}$$

Of course, if $\tau_1 = \tau_2$ or $\omega_1 = \omega_2$, then $\det Q = 0$, but we assume that they are not equal. Since $\det Q$ is a real analytic function of τ_1 and τ_2 then for an open and dense set of values of (τ_1, τ_2), the determinant is nonzero. □

3.3. $\mathbf{Z}_2$-symmetric first-order scalar equation with two delays

In this section, we consider the restrictions on Eq. (15). We rewrite the equation as (2) with $\mu = 0$. Since the equation is $\mathbf{Z}_2$ symmetric there are no even degree terms in the Taylor expansion of F. We now determine the normal form (19) on the centre manifold. The homogeneous polynomial of degree three in the Taylor expansion of F is

$$F_3(u_1, u_2) = a_{30}u_1^3 + a_{21}u_1^2u_2 + a_{12}u_1u_2^2 + a_{03}u_2^3,$$

and $F_2 \equiv 0$, thus dropping the conjugate equations and setting $y = 0$ we obtain the system

$$\begin{aligned} \dot{z}_1 &= i\omega_1 z_1 + \psi_1(0)F_3(\Phi z), \\ \dot{z}_2 &= i\omega_2 z_2 + \psi_3(0)F_3(\Phi z). \end{aligned} \tag{23}$$

After normal form transformation of the cubic terms, we are left with Eq. (9) where by Proposition 2.1

$$\begin{aligned} c_{11} &= \frac{1}{2}\frac{\partial^3\psi_1(0)F_3(\Phi z)}{\partial z_1^2\partial\bar{z}_1}\bigg|_{z=0}, & c_{12} &= \frac{\partial^3\psi_1(0)F_3(\Phi z)}{\partial z_1\partial z_2\partial\bar{z}_2}\bigg|_{z=0}, \\ c_{22} &= \frac{1}{2}\frac{\partial^3\psi_3(0)F_3(\Phi z)}{\partial z_2^2\partial\bar{z}_2}\bigg|_{z=0}, & c_{21} &= \frac{\partial^3\psi_3(0)F_3(\Phi z)}{\partial z_2\partial z_1\partial\bar{z}_1}\bigg|_{z=0}. \end{aligned} \tag{24}$$

The genericity result for the $\mathbf{Z}_2$-symmetric equation is the following.

Proposition 3.3. *Suppose that the* $\mathbf{Z}_2$*-symmetric equation* (15) *has a nonresonant double Hopf bifurcation at* 0. *Then, generically, there are no restrictions on the values that the coefficients* $(\mathrm{Re}(c_{11}), \mathrm{Re}(c_{12}), \mathrm{Re}(c_{22}), \mathrm{Re}(c_{21}))$ *can take in* (10).

Proof. We compute explicitly the terms of degree three:

$$\begin{aligned}F_3(\Phi(-\tau_1)z, \Phi(-\tau_2)z) =\ & \eta(\omega_1)z_1^3 + \eta(-\omega_1)\bar{z}_1^3 + \eta(\omega_2)z_2^3 + \eta(-\omega_2)\bar{z}_2^3 + \zeta(\omega_1)z_1^2\bar{z}_1 \\ & + \zeta(-\omega_1)\bar{z}_1^2 z_1 + \zeta(\omega_2)z_2^2\bar{z}_2 + \zeta(-\omega_2)\bar{z}_2^2 z_2 + \xi(\omega_1,\omega_2)z_1^2 z_2 \\ & + \xi(\omega_1,-\omega_2)z_1^2\bar{z}_2 + \xi(-\omega_1,\omega_2)\bar{z}_1^2 z_2 + \xi(-\omega_1,-\omega_2)\bar{z}_1^2\bar{z}_2 \\ & + \xi(\omega_2,\omega_1)z_2^2 z_1 + \xi(-\omega_2,\omega_1)\bar{z}_2^2 z_1 + \xi(-\omega_2,-\omega_1)\bar{z}_2^2\bar{z}_1 \\ & + \xi(\omega_2,-\omega_1)\bar{z}_1^2 z_2 + v(\omega_1,\omega_2)z_1\bar{z}_1 z_2 + v(\omega_1,-\omega_2)z_1\bar{z}_1\bar{z}_2 \\ & + v(\omega_2,\omega_1)z_2\bar{z}_2 z_1 + v(\omega_2,-\omega_1)z_2\bar{z}_2\bar{z}_1, \end{aligned} \tag{25}$$

where

$$\eta(u) = a_{30}e^{-3iu\tau_1} + a_{21}e^{-(2\tau_1+\tau_2)iu} + a_{12}e^{-(\tau_1+2\tau_2)iu} + a_{03}e^{3iu\tau_2},$$

$$\zeta(u) = 3a_{30}e^{-iu\tau_1} + a_{21}(2e^{-iu\tau_2} + e^{(-2\tau_1+\tau_2)iu}) + a_{12}(e^{(-2\tau_2+\tau_1)iu} + 2e^{-iu\tau_1}) + 3a_{03}e^{-iu\tau_2},$$

$$\begin{aligned}\xi(u,v) =\ & 3a_{30}e^{-i\tau_1(2u+v)} + a_{21}e^{-iu\tau_1}(2e^{-i(v\tau_1+u\tau_2)} + e^{-i(u\tau_1+v\tau_2)}) \\ & + a_{12}e^{-iu\tau_2}(e^{-i(v\tau_1+u\tau_2)} + 2e^{-i(u\tau_1+v\tau_2)}) + 3a_{03}e^{-i\tau_2(2u+v)},\end{aligned}$$

$$\begin{aligned}v(u,v) =\ & 6a_{30}e^{-iv\tau_1} + 2a_{21}(e^{-i(v\tau_1+u\tau_1-u\tau_2)} + e^{-iv\tau_2} + e^{-i(v\tau_1-u\tau_1+u\tau_2)}) \\ & + 2a_{12}(e^{-i(v\tau_2+u\tau_1-u\tau_2)} + e^{-iv\tau_1} + e^{-i(v\tau_2+u\tau_2-u\tau_1)}) + 6a_{03}e^{-iv\tau_2}.\end{aligned}$$

Using (24) and (25), we compute $(c_{11}, c_{12}, c_{22}, c_{21})$ explicitly:

$$c_{11} = \psi_1(0)\zeta(\omega_1), \qquad c_{12} = \psi_1(0)v(\omega_2,\omega_1),$$

$$c_{22} = \psi_3(0)\zeta(\omega_2), \qquad c_{21} = \psi_3(0)v(\omega_1,\omega_2).$$

We now show that generically $(\mathrm{Re}(c_{11}), \mathrm{Re}(c_{22}), \mathrm{Re}(c_{12}), \mathrm{Re}(c_{21}))$ can take arbitrary values. Consider $(\mathrm{Re}(c_{11}), \mathrm{Re}(c_{22}), \mathrm{Re}(c_{12}), \mathrm{Re}(c_{21}))$ as a linear system in $(a_{30}, a_{21}, a_{12}, a_{03})$. After tedious computations, one can show that the matrix of coefficients of $(a_{30}, a_{21}, a_{12}, a_{03})$ is

$$\begin{bmatrix} 3\alpha V_1 & 3\alpha V_2 & 6\alpha V_1 & 6\alpha V_2 \\ \alpha(V_3+2V_1c(\omega_1)) & \alpha(V_4+2V_2c(\omega_2)) & 2\alpha(V_3+2V_1c(\omega_2)) & 2\alpha(V_4+2V_2c(\omega_1)) \\ \alpha(V_1+2V_3c(\omega_1)) & \alpha(V_2+2V_4c(\omega_2)) & 2\alpha(V_1+2V_3c(\omega_2)) & 2\alpha(V_2+2V_4c(\omega_1)) \\ 3\alpha V_3 & 3\alpha V_4 & 6\alpha V_3 & 6\alpha V_4 \end{bmatrix}, \tag{26}$$

where $V_1 = \cos(-\beta(\omega_1) + \omega_1\tau_1)$, $V_2 = \cos(-\beta(\omega_2) + \omega_2\tau_1)$, $V_3 = \cos(-\beta(\omega_1) + \omega_1\tau_2)$, $V_4 = \cos(-\beta(\omega_2) + \omega_2\tau_2)$, $c(u) = \cos(u(\tau_1 - \tau_2))$, $\alpha = |\psi_1(0)|$, $\beta(\omega_1) = \arg(\psi_1(0))$ and $\beta(\omega_2) = \arg(\psi_3(0))$. The determinant of this matrix is

$$-144\alpha^4(\cos(\omega_1(\tau_1-\tau_2)) - \cos(\omega_2(\tau_1-\tau_2)))^2(V_2V_3 - V_1V_4)^2.$$

Suppose that $\tau_1 \neq \tau_2$ and $\omega_1(\tau_1-\tau_2) \neq \omega_2(\tau_1-\tau_2) + 2k\pi$ for all $k \in \mathbf{Z}$, then the determinant vanishes if and only if $V_2V_3 - V_1V_4 = 0$.

At a nonresonant double Hopf bifurcation point, Bélair and Campbell [2] show that

$$a_{01}\cos(\omega_1\tau_2) = -a_{10}\cos(\omega_1\tau_1), \qquad a_{01}\cos(\omega_2\tau_2) = -a_{10}\cos(\omega_2\tau_1),$$

$$\begin{aligned} a_{01}\sin(\omega_1\tau_2) &= a_{10}\omega_1 - a_{10}\sin(\omega_1\tau_1), \\ a_{01}\sin(\omega_2\tau_2) &= a_{10}\omega_2 - a_{10}\sin(\omega_2\tau_1). \end{aligned} \tag{27}$$

Hence $V_2V_3 - V_1V_4$ simplifies to a real analytic function of τ_1

$$\begin{aligned} &\frac{a_{10}}{a_{01}}((\sin(\omega_2\tau_1)\omega_1 - \sin(\omega_1\tau_1)\omega_2)\sin(\beta(\omega_1))\sin(\beta(\omega_2)) + \omega_1\cos(\omega_2\tau_1)\sin(\beta(\omega_1)) \\ &\quad \times \cos(\beta(\omega_2)) - \omega_2\cos(\omega_1\tau_1)\cos(\beta(\omega_1))\sin(\beta(\omega_2))). \end{aligned}$$

Since the zeroes of nonzero analytic functions are isolated, then for an open and dense set of values of τ_1, we have that $V_2V_3 - V_1V_4 \neq 0$. Hence, generically, there are no restrictions on the cubic coefficients of the normal form. □

Bélair and Campbell [2] compute the normal form at a double Hopf bifurcation to cubic order for the delay differential equation

$$\dot{x}(t) = -A_1\tanh(x(t-T_1)) - A_2\tanh(x(t-T_2)). \tag{28}$$

Eq. (28) is $\mathbf{Z}_2$ symmetric with $a_{21} = 0$ and $a_{12} = 0$. They show that there are relations between the coefficients of the cubic monomials of the normal form. Therefore, not all possible phase portraits in a neighbourhood of the origin in parameter space are realized near the double Hopf bifurcation point. We recover their result.

Corollary 3.4. *Suppose that* $F_3(u_1, u_2) = a_{30}u_1^3 + a_{03}u_2^3$. *Then*

$$\mathrm{Re}(c_{12}) = 2\mathrm{Re}(c_{11}) \quad \textit{and} \quad \mathrm{Re}(c_{21}) = 2\mathrm{Re}(c_{22})$$

where

$$\begin{aligned}\mathrm{Re}(c_{11}) = &\frac{3\mathrm{Re}(\psi_1(0))}{a_{01}}\cos(\omega_1\tau_1)(a_{30}a_{01} - a_{03}a_{10}) + \frac{3\mathrm{Im}(\psi_1(0))}{a_{01}} \\ &\times [(a_{30}a_{01} - a_{03}a_{10})\sin(\omega_1\tau_1) + a_{03}a_{10}\omega_1],\end{aligned}$$

and

$$\begin{aligned}\mathrm{Re}(c_{22}) = &\frac{3\mathrm{Re}(\psi_3(0))}{a_{01}}\cos(\omega_2\tau_1)(a_{30}a_{01} - a_{03}a_{10}) + \frac{3\mathrm{Im}(\psi_3(0))}{a_{01}} \\ &\times [(a_{30}a_{01} - a_{03}a_{10})\sin(\omega_2\tau_1) + a_{03}a_{10}\omega_2].\end{aligned}$$

Moreover, if $\mathrm{Re}(c_{11}) \neq 0$ *and* $\mathrm{Re}(c_{22}) \neq 0$, *then the double Hopf bifurcation is determined to third order.*

Proof. Set $a_{12} = a_{21} = 0$ in $(\mathrm{Re}(c_{11}), \mathrm{Re}(c_{22}), \mathrm{Re}(c_{12}), \mathrm{Re}(c_{21}))$ to obtain the result. Then use conditions (27). Now, $\mathrm{Re}(c_{11})\mathrm{Re}(c_{22}) - \mathrm{Re}(c_{12})\mathrm{Re}(c_{21}) = -3\mathrm{Re}(c_{11})\mathrm{Re}(c_{22})$. Thus the nondegeneracy conditions for the vector field to be determined to third order are satisfied if $\mathrm{Re}(c_{11}) \neq 0$ and $\mathrm{Re}(c_{22}) \neq 0$. □

We now discuss the possible restrictions on the phase portraits near the nonresonant double Hopf bifurcation point. We rewrite system (13) as in Guckhenheimer [12]

$$\dot{r}_1 = r_1(\mu_1 + r_1^2 + br_2^2),$$

$$\dot{r}_2 = r_2(\mu_2 + cr_1^2 + dr_2^2), \tag{29}$$

where $d = \mathrm{Re}(c_{22})/|\mathrm{Re}(c_{22})| = \pm 1$, $c = \mathrm{Re}(c_{21})/|\mathrm{Re}(c_{11})|$ and $b = \mathrm{Re}(c_{12})/|\mathrm{Re}(c_{22})|$. In Table 1, we reproduce Table 7.5.2 of [12] which shows the 12 unfolding cases for the nonresonant double Hopf bifurcation.

Corollary 3.4 implies that sgn d = sgn c. Table 1 shows that the unfoldings II, IVa, IVb, V, VIIa and VIIb are not possible in this case.

3.4. First-order scalar equation with two delays

For the general scalar delay equation, the calculation of the cubic normal form requires lengthy calculations and the size of the expressions for the coefficients of the cubic terms become quickly unmanageable. Instead, we use Proposition 3.3 to obtain a similar result for general scalar equations.

Proposition 3.5. *Suppose that the scalar delay-differential equation* (14) *has a nonresonant double Hopf bifurcation at* 0. *Then, generically, there are no restrictions on the values that the coefficients* $(\mathrm{Re}(c_{11}), \mathrm{Re}(c_{12}), \mathrm{Re}(c_{22}), \mathrm{Re}(c_{21}))$ *can take in* (10).

Proof. We remove the quadratic terms in (19) using the normal form transformation

$$(z, y) = (\tilde{z}, \tilde{y}) + (\sigma_1^2(\tilde{z}), \sigma_2^2(\tilde{z})).$$

Dropping the $\sim$ symbol on z, the cubic degree terms in the normal form on the centre manifold are obtained by multiplying the following expression by $\Psi(0)$:

$$F_3(\Phi z) + (d_z F_2(\Phi z))\sigma_1^2(z) + (d_y F_2(\Phi z))\sigma_2^2(z). \tag{30}$$

The coefficients $(c_{11}, c_{12}, c_{22}, c_{21})$ are functions of the coefficients (a_{20}, a_{11}, a_{02}) of F_2 and $(a_{30}, a_{21}, a_{12}, a_{03})$ of F_3. Let T be matrix (26),

$$C = \begin{bmatrix} \mathrm{Re}(c_{11}) \\ \mathrm{Re}(c_{12}) \\ \mathrm{Re}(c_{22}) \\ \mathrm{Re}(c_{21}) \end{bmatrix} \quad \text{and} \quad C_3 = \begin{bmatrix} a_{30} \\ a_{21} \\ a_{12} \\ a_{30} \end{bmatrix}.$$

From (24) and (30), we see that the coefficients of the cubic terms can be written as

$$C = TC_3 + R(a_{20}, a_{11}, a_{02}), \tag{31}$$

where $R(a_{20}, a_{11}, a_{02})$ is a vector in $\mathbf{R}^4$. Hence, for any $C \in \mathbf{R}^4$ and coefficients (a_{20}, a_{11}, a_{02}), by Proposition 3.3, generically, we can find C_3 such that Eq. (31) is satisfied. $\square$

3.5. First-order scalar equation with one delay

The quadratic and cubic nonlinearities are

$$F_2(u) = a_2 u \quad \text{and} \quad F_3(u) = a_3 u.$$

Table 1
The 12 unfolding cases of (29)

Case	Ia	Ib	II	III	IVa	IVb	V	VIa	VIb	VIIa	VIIb	VIII
d	+1	+1	+1	+1	+1	+1	−1	−1	−1	−1	−1	−1
b	+	+	+	−	−	−	+	+	+	−	−	−
c	+	+	−	+	−	−	+	−	−	+	+	−
$d-bc$	+	−	(+)	(+)	+	−	(−)	+	−	+	−	(−)

In Faria and Magalhães [6], it is shown that the homogeneous polynomials $g_i^2(x,y)$ of (7) are given by

$$g_i^2(x,y) = \tilde{f}_i^2(x,y) - [D_x U_j^2(x)Bx - A_{Q^1}(U_j^2(x))],$$

where U_j^2 is the nonlinear part of the normal form transformation and $\tilde{f}_i^2$ denote the terms of degree i obtained after normal form computations to degree $i-1$. Thus, because of assumptions H1 and H2 the polynomial U_j^2 is determined by solving

$$D_x U_j^2(x)Bx - A_{Q^1}(U_j^2(x)) = \tilde{f}_i^2(x,0). \tag{32}$$

Note that $\tilde{f}_2^2(x,0) = f_2{}^2(x,0)$.

In our case, let $(\sigma_1^2(z), \sigma_2^2(z))$ be the nonlinear part of the normal form transformation for quadratic polynomials where

$$\sigma_2^2(z)(\theta) = \sum_{|q|=2} h_{q_1,q_2,q_3,q_4}(\theta) z_1^{q_1}\,\bar{z}_1^{q_2}\,z_2^{q_3}\,\bar{z}_2^{q_4},$$

with $|q| = q_1 + q_2 + q_3 + q_4$ and $h_{q_1,q_2,q_3,q_4}(\theta) \in Q^1$. Then (32) becomes

$$-\omega_1\left[\frac{\partial\sigma_2^2}{\partial z_1}\bar{z}_1 - \frac{\partial\sigma_2^2}{\partial\bar{z}_1}z_1\right] - \omega_2\left[\frac{\partial\sigma_2^2}{\partial z_2}\bar{z}_2 - \frac{\partial\sigma_2^2}{\partial\bar{z}_2}z_2\right] - \dot{\sigma}_2^2(z) = -\Phi\Psi(0)a_2(\Phi(-\tau)z)^2 \tag{33}$$

with boundary conditions

$$\dot{\sigma}_2^2(z)(0) - L(\sigma_2^2(z)) = a_2(\Phi(-\tau)z)^2.$$

A rough expression for the normal form transformation of the quadratic polynomial of the $\dot{y}$ equation is given here.

Proposition 3.6.

$$\begin{aligned}\sigma_2^2(z)(\theta) = {} & a_2(P_1(\theta,\omega_1,\omega_2)z_1^2 + P_1(\theta,-\omega_1,\omega_2)\bar{z}_1^2 + P_1(\theta,\omega_2,\omega_1)z_2^2 + P_1(\theta,-\omega_2,\omega_1)\bar{z}_2^2 \\ & + P_2(\theta,\omega_1,\omega_2)z_1\bar{z}_1 + P_2(\theta,\omega_2,\omega_1)z_2\bar{z}_2 + Q_1(\theta,\omega_1,\omega_2)z_1z_2 \\ & + Q_1(\theta,-\omega_1,-\omega_2)\bar{z}_1\bar{z}_2 + Q_2(\theta,\omega_1,\omega_2)\bar{z}_1z_2 + Q_2(\theta,-\omega_1,-\omega_2)z_1\bar{z}_2),\end{aligned}$$

where P_1, P_2, Q_1 and Q_2 are smooth functions of θ, ω_1 and ω_2.

The proof of Proposition 3.6 is found in the appendix. We now give expressions for the cubic coefficients of the normal form.

Proposition 3.7. *The coefficients of the cubic terms in the normal form are given below*:

$$\begin{aligned}\mathrm{Re}(c_{11}) = {} & 3a_3\mathrm{Re}(\psi_1(0)e^{i\omega_1\tau}) + 2a_2^2\big[2\omega_1^{-1}\mathrm{Re}(\psi_1(0)e^{i\omega_1\tau})\mathrm{Im}(\psi_1(0)e^{i\omega_1\tau})\\ & - 4\omega_2^{-1}\mathrm{Re}(\psi_1(0)e^{i\omega_1\tau})\mathrm{Im}(\psi_3(0)e^{i\omega_2\tau}) + (4\omega_1^2-\omega_2^2)^{-1}\\ & \times (2\omega_1\mathrm{Re}(\psi_3(0)e^{i\omega_2\tau})\mathrm{Im}(\psi_1(0)e^{i\omega_1\tau}) + \omega_2\mathrm{Im}(\psi_3(0)e^{i\omega_2\tau})\mathrm{Re}(\psi_1(0)e^{i\omega_1\tau}))\big]\\ & + 2a_2^2\mathrm{Re}[\psi_1(0)(e^{-i\omega_1\tau}P_2(-\tau,\omega_1,\omega_2) + e^{i\omega_1\tau}P_1(-\tau,\omega_1,\omega_2))],\end{aligned}$$

$$\begin{aligned}\mathrm{Re}(c_{12}) = {} & 6a_3\mathrm{Re}(\psi_1(0)e^{i\omega_1\tau}) + 4a_2^2\Big[2\omega_1^{-1}(\mathrm{Re}(\psi_1(0)e^{i\omega_1\tau})\mathrm{Im}(\overline{\psi_1(0)}e^{-i\omega_2\tau})\\ & + \mathrm{Im}(\psi_1(0)e^{i\omega_1\tau})\mathrm{Re}(\psi_3(0)e^{i\omega_2\tau})) + 2\omega_2^{-1}\mathrm{Im}(\overline{\psi_3(0)}e^{i\omega_2\tau})\mathrm{Re}(\psi_1(0)e^{i\omega_1\tau})\\ & + 2(\omega_1^2-4\omega_2^2)^{-1}(\omega_1\mathrm{Re}(\psi_3(0)e^{i\omega_2\tau})\mathrm{Im}(\psi_1(0)e^{i\omega_1\tau})\\ & -2\omega_2\mathrm{Re}(\psi_1(0)e^{i\omega_1\tau})\mathrm{Im}(\psi_3(0)e^{i\omega_2\tau}))\Big]\\ & + 2a_2^2\mathrm{Re}[\psi_1(0)(e^{-i\omega_1\tau}P_2(-\tau,\omega_2,\omega_1)\\ & + e^{-i\omega_2\tau}Q_2(-\tau,-\omega_1,-\omega_2) + e^{i\omega_2\tau}Q_1(-\tau,\omega_1,\omega_2))].\end{aligned}$$

Letting $c_{11} = c_{11}(\omega_1,\omega_2)$ *and* $c_{12} = c_{12}(\omega_1,\omega_2)$ *then* $\mathrm{Re}(c_{22}) = \mathrm{Re}(c_{11}(\omega_2,\omega_1))$ *and* $\mathrm{Re}(c_{21}) = \mathrm{Re}(c_{12}(\omega_2,\omega_1))$.

Proof. Recall first that $\psi_1(0) = \varepsilon(\omega_1)$ and $\psi_3(0) = \varepsilon(\omega_2)$ for some function ε. The quadratic and cubic polynomials are given below:

$$\begin{aligned}F_2(\Phi(-\tau)z) = {} & a_2(e^{2i\omega_1\tau}z_1^2 + e^{-2i\omega_1\tau}\bar{z}_1^2 + e^{2i\omega_2\tau}z_2^2 + e^{-2i\omega_2\tau}\bar{z}_2^2 + 2z_1\bar{z}_1 + 2e^{i\tau(\omega_1+\omega_2)}z_1z_2\\ & + 2e^{i\tau(\omega_1-\omega_2)}z_1\bar{z}_2 + 2e^{-i\tau(\omega_1-\omega_2)}\bar{z}_1z_2 + 2e^{-i\tau(\omega_1+\omega_2)}\bar{z}_1\bar{z}_2 + 2z_2\bar{z}_2),\end{aligned}$$

$$\begin{aligned}F_3(\Phi(-\tau)z) = {} & a_3(e^{3i\omega_1\tau}z_1^3 + 3e^{i\omega_1\tau}z_1^2\bar{z}_1 + 3e^{i\tau(\omega_2+2\omega_1)}z_1^2z_2 + 3e^{-i\tau(\omega_2-2\omega_1)}\bar{z}_2z_1^2\\ & + 3e^{-i\omega_1\tau}z_1\bar{z}_1^2 + 6e^{i\omega_2\tau}z_1\bar{z}_1z_2 + 6e^{-i\omega_2\tau}\bar{z}_2z_1\bar{z}_1 + 3e^{i\tau(2\omega_2+\omega_1)}z_1z_2^2\\ & + 6e^{i\omega_1\tau}\bar{z}_2z_1z_2 + 3e^{-i\tau(2\omega_2+\omega_1)}\bar{z}_2^2z_1 + e^{-3i\omega_1\tau}\bar{z}_1^3 + 3e^{i\tau(\omega_2-2\omega_1)}\bar{z}_1^2z_2\\ & + 3e^{-i\tau(\omega_2+2\omega_1)}\bar{z}_2\bar{z}_1^2 + 3e^{i\tau(2\omega_2-\omega_1)}\bar{z}_1z_2^2 + 6e^{-i\omega_1\tau}\bar{z}_2\bar{z}_1z_2\\ & + 3e^{-i\tau(2\omega_2+\omega_1)}\bar{z}_2^2\bar{z}_1 + e^{3i\omega_2\tau}z_2^3 + 3e^{i\omega_2\tau}\bar{z}_2z_2^2\\ & + 3e^{-i\omega_2\tau}\bar{z}_2^2z_2 + e^{-3i\omega_2\tau}\bar{z}_2^3).\end{aligned}$$

We perform the computations for the system in complex coordinates and then take the appropriate real parts. Eq. (30) gives the cubic terms after normal form transformation of the quadratic terms. The part of the coefficients c_{ij} $(i,j=1,2)$ coming from

$$F_3(\Phi(-\tau)z)+(d_zF_2(\Phi(-\tau)z))\sigma_1^2(z)$$

are found using the result of Knobloch [15] on the computation of the cubic normal form for ODEs. The remaining part of the coefficients is computed from

$$d_y(F_2(\Phi(-\tau)z+y))|_{y=0}\sigma_2^2=2a_2\Phi(-\tau)z\sigma_2^2(z). \tag{34}$$

Thus,

$$\begin{aligned}c_{11}={}&3\psi_1(0)a_3e^{i\omega_1\tau}+2a_2^2\psi_1(0)e^{i\omega_1\tau}\left[\frac{-1}{i\omega_1}\left(2i\mathrm{Im}(\psi_1(0)e^{i\omega_1\tau})-\frac{2}{3}\overline{\psi_1(0)}e^{-i\omega_1\tau}\right)\right.\\&\left.-\frac{4}{i\omega_2}i\mathrm{Im}(\psi_3(0)e^{i\tau\omega_2})-\frac{i}{4\omega_1^2-\omega_2^2}\left(2\omega_1\mathrm{Re}(\psi_3(0)e^{i\tau\omega_2})+\omega_2 i\mathrm{Im}(\psi_3(0)e^{i\tau\omega_2})\right)\right]\\&+\ 2a_2^2\psi_1(0)[e^{-i\omega_1\tau}h_{1,1,0,0}(-\tau)+e^{i\omega_1\tau}h_{2,0,0,0}(-\tau)],\end{aligned}$$

$$\begin{aligned}c_{12}={}&6\psi_1(0)a_3e^{i\omega_1\tau}+4a_2^2\psi_1(0)e^{i\omega_1\tau}\left[2\omega_1^{-1}(\mathrm{Im}(\overline{\psi_1(0)}e^{-i\omega_1\tau})-i\mathrm{Re}(\psi_3(0)e^{i\omega_2\tau}))\right.\\&+\ 2\omega_2^{-1}\mathrm{Im}(\overline{\psi_3(0)}e^{i\omega_2\tau})+\frac{\overline{\psi_1(0)}e^{-i\omega_1\tau}}{i(2\omega_1-\omega_2)}+\frac{\overline{\psi_1(0)}e^{-i\omega_1\tau}}{i(2\omega_1+\omega_2)}\\&\left.-\frac{2i}{\omega_1^2-4\omega_2^2}(\omega_1\mathrm{Re}(\psi_3(0)e^{i\omega_2\tau})+2\omega_2 i\mathrm{Im}(\psi_3(0)e^{i\omega_2\tau}))\right]\\&+\ 2a_2^2\psi_1(0)[e^{-i\omega_1\tau}h_{0,0,1,1}(-\tau)+e^{-i\omega_2\tau}h_{1,0,0,1}(-\tau)+e^{i\omega_2\tau}h_{1,0,1,0}(-\tau)].\end{aligned}$$

It is a straightforward computation using formulae (11a) and (11b) of Knobloch [15], (34) and Proposition 3.6 to verify that $c_{22}=c_{11}(\omega_2,\omega_1)$ and $c_{21}=c_{12}(\omega_2,\omega_1)$. Taking the real parts yields the result. $\square$

Corollary 3.8. *If* $a_2=0$ *then* $\mathrm{Re}(c_{12})=2\mathrm{Re}(c_{11})=6a_3\mathrm{Re}(\psi_1(0)e^{i\omega_1\tau})$ *and* $\mathrm{Re}(c_{21})=2\mathrm{Re}(c_{22})=6a_3\mathrm{Re}(\psi_3(0)e^{i\omega_2\tau})$. *As in Corollary* 3.4, *the double Hopf bifurcation is determined to third order if* $\mathrm{Re}(c_{11})\neq 0$ *and* $\mathrm{Re}(c_{22})\neq 0$.

If $a_2=0$, since $\mathrm{Re}(c_{12})=2\mathrm{Re}(c_{11})$ and $\mathrm{Re}(c_{21})=2\mathrm{Re}(c_{22})$, the restrictions on the possible phase portraits near a double Hopf point are similar to the restrictions stated after Corollary 3.4.

Now, letting $a_2\neq 0$, a priori many more unfolding cases are possible since $\mathrm{sgn}\, d$ and $\mathrm{sgn}\, c$ need not be equal anymore. However, we now show that there always are restrictions on the possible flows near the double Hopf point for fixed values of ω_1,ω_2 and τ. Before we state the result, we perform some transformations on the expressions for the cubic coefficients. From Proposition 3.7 the coefficients in the normal form can be written as

$$\mathrm{Re}(c_{11}) = p_1 a_3 + p_2 a_2^2, \qquad \mathrm{Re}(c_{12}) = q_1 a_3 + q_2 a_2^2,$$

$$\mathrm{Re}(c_{21}) = r_1 a_3 + r_2 a_2^2, \qquad \mathrm{Re}(c_{22}) = s_1 a_3 + s_2 a_2^2,$$

where $p_1, p_2, q_1, q_2, r_1, r_2, s_1, s_2$ are constants since the calculation is made for ω_1, ω_2 and τ fixed. Now, if the determinant of $M = \begin{bmatrix} p_1 & s_1 \\ p_2 & s_2 \end{bmatrix}$ is nonzero, we can write

$$\mathrm{Re}(c_{12}) = \gamma_1 \mathrm{Re}(c_{11}) + \gamma_2 \mathrm{Re}(c_{22}) \quad \text{and} \quad \mathrm{Re}(c_{21}) = \delta_1 \mathrm{Re}(c_{11}) + \delta_2 \mathrm{Re}(c_{22}),$$

where $(\gamma_1, \gamma_2)^t = M^{-1}(q_1, q_2)^t$ and $(\delta_1, \delta_2)^t = M^{-1}(r_1, r_2)^t$. Hence,

$$b = \frac{\mathrm{Re}(c_{12})}{|\mathrm{Re}(c_{22})|} = \gamma_1 \frac{\mathrm{Re}(c_{11})}{|\mathrm{Re}(c_{22})|} \pm \gamma_2 \quad \text{and} \quad c = \frac{\mathrm{Re}(c_{21})}{|\mathrm{Re}(c_{11})|} = \pm \delta_1 + \delta_2 \frac{\mathrm{Re}(c_{22})}{|\mathrm{Re}(c_{11})|}. \tag{35}$$

We now state the result. Note that the proof of the proposition also gives a method to determine which restrictions occurs for a particular system.

Proposition 3.9. *Assume the nondegeneracy condition* $\det M \neq 0$ *is satisfied. Then there always are restrictions on the possible flows of system* (16) *near a nonresonant double Hopf bifurcation point.*

Proof. We need to show that for all values of a_2 and a_3, there are some combinations of signs of b, c and d which are prohibited so that some of the unfoldings of Table 1 cannot occur. The equations $\mathrm{Re}(c_{11}) = p_1 a_3 + p_2 a_2^2 = 0$ and $\mathrm{Re}(c_{22}) = s_1 a_3 + s_2 a_2^2 = 0$ define two parabolae passing through $(0,0)$ in (a_2, a_3) space. By the nondegeneracy condition $\det M \neq 0$ the two parabolae may not coincide.

The parabola $s_1 a_3 + s_2 a_2^2 = 0$ separates the plane (a_2, a_3) in two connected regions R_1, R_2. Let R_1 be the region which does not contain the parabola $p_1 a_3 + p_2 a_2^2 = 0$. Thus in R_1, $\operatorname{sgn} \mathrm{Re}(c_{22})$ and $\operatorname{sgn} \mathrm{Re}(c_{11})$ are constant. Therefore, we can choose values (a_2, a_3) such that one case of sign of $\mathrm{Re}(c_{22})$ forces the sign of $\mathrm{Re}(c_{11})$.

Let $\Delta = \mathrm{Re}(c_{11})/\mathrm{Re}(c_{22})$ and rewrite

$$b = \operatorname{sgn} \mathrm{Re}(c_{22})(\gamma_1 \Delta + \gamma_2), \qquad c = \operatorname{sgn} \mathrm{Re}(c_{11})(\delta_1 + \delta_2/\Delta),$$

where Δ is considered as a variable.

Now the positions of the parabolae is determined by $-p_2/p_1$, $-s_2/s_1$ and the signs of p_1 and s_1. Consider the cases of positions of parabolae where $d = \operatorname{sgn} \mathrm{Re}(c_{22}) = +1$ forces $\operatorname{sgn} \mathrm{Re}(c_{11})$ to be constant. This is the case for instance, if $-s2/s1 > -p_2/p_1$ and p_1, s_1 are positive, then $\mathrm{Re}(c_{22}) > 0$ forces $\mathrm{Re}(c_{11}) > 0$. In such a situation, Δ takes its values either in $(-\infty, 0)$ or $(0, \infty)$. In either case, b and c are monotone functions of Δ on its interval of definition. Therefore, there are at most three intervals where b and c have constant signs. Hence, there is always a choice of signs of b and c which does not occur.

A similar argument holds for the cases of positions of the parabolae where $d = \operatorname{sgn}\operatorname{Re}(c_{22}) = -1$ forces $\operatorname{sgn}\operatorname{Re}(c_{11})$ to be constant. This completes the proof. □

4. nth-order scalar equation with delayed feedback, $n \geqslant 2$

Consider now the nth-order delay-differential equation ($n \geqslant 2$)

$$u^{(n)} + \beta_1 u^{(n-1)} + \cdots + \beta_n u = f(u(t-\tau)), \tag{36}$$

where $f(0) = 0$, β_j ($j = 1, \ldots, n$) are constants and τ is the time delay. This equation generalizes the harmonic oscillator with delayed feedback

$$\ddot{u} + \beta_1 \dot{u} + \beta_2 u = f(u(t-\tau)) \tag{37}$$

studied by Campbell et al. [4].

In this section, we prove the following unfolding result for Eq. (36).

Theorem 4.1. *Suppose that* (36) *has a nonresonant double Hopf bifurcation point at the origin. Then, generically, the two parameter family of delay-differential equations*

$$u^{(n)} + \beta_1 u^{(n-1)} + \cdots + (\beta_n + v_1)u = (a_1 + v_2)u(t-\tau) + o(u(t-\tau)) \tag{38}$$

provides a universal unfolding for the double Hopf bifurcation. However, generically, there always are restrictions on the possible flows of (36) *near a double Hopf bifurcation point.*

The proof of Theorem 4.1 is given by Lemma 4.2 and Proposition 4.4.

4.1. The $C = P \oplus Q$ decomposition

Truncate f to degree three in its Taylor expansion

$$f(u(t-\tau)) = a_1 u(t-\tau) + a_2 u^2(t-\tau) + a_3 u^3(t-\tau)$$

and rewrite (36) as a system of n first-order delay-differential equations

$$\begin{cases} \dot{u} = v_1, \\ \dot{v}_1 = v_2, \\ \quad \vdots \\ \dot{v}_{n-1} = -\beta_1 v_{n-1} - \cdots - \beta_n u + a_1 u(t-\tau) + a_2 u^2(t-\tau) + a_3 u^3(t-\tau). \end{cases} \tag{39}$$

So,

$$L(u_t, v_t) = \begin{bmatrix} v_1 \\ \vdots \\ v_{n-2} \\ -\beta_1 v_{n-1} - \cdots - \beta_n v_1 + a_1 u(t-\tau) \end{bmatrix},$$

$$F(u_t) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_2 u^2(t-\tau) + a_3 u^3(t-\tau). \end{bmatrix}.$$

At a double Hopf point, the basis of P is given by the columns of $\Phi = [\Phi_1, \dots, \Phi_n]^t$ where

$$\Phi_j = ((i\omega_1)^{j-1} e^{i\omega_1\theta}, (-i\omega_1)^{j-1} e^{-i\omega_1\theta}, (i\omega_2)^{j-1} e^{i\omega_2\theta}, (-i\omega_2)^{j-1} e^{-i\omega_2\theta}).$$

The basis of the adjoint problem is given by the rows of $\Psi = [\Psi_1, \dots, \Psi_n]$ with $\Psi_j = (\Psi_j^1, \Psi_j^2, \Psi_j^3, \Psi_j^4)^t$ where $\Psi = (\Phi^t, \Phi)^{-1}\Phi^t$ and $(\cdot,\cdot)$ is the bilinear form (4).

Let $(u, v_1, \dots, v_{n-1})^t = \Phi z + y$ where $y = (y_1, \dots, y_n)^t \in Q \cap C^1([-\tau, 0], \mathbf{R}^n)$. We rewrite (39) as an infinite dimensional system. Note that F is only function of $u = \Phi_1(-\tau)z + y_1$, thus

$$\begin{aligned} \dot z &= Bz + \Psi(0) F(\Phi_1(-\tau)z + y_1), \\ \dot y &= A_{Q^1} y + (I - \pi) X_0 F(\Phi_1(-\tau)z + y_1), \end{aligned} \tag{40}$$

where B is (8). Now,

$$F(\Phi_1(-\tau)z + y_1) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_2(\Phi_1(-\tau)z + y_1)^2 + a_3(\Phi_1(-\tau)z + y_1)^3 \end{bmatrix}.$$

Hence (40) becomes

$$\begin{aligned} \dot z &= Bz + \Psi_n(0)(a_2(\Phi_1(-\tau)z + y_1)^2 + a_3(\Phi_1(-\tau)z + y_1)^3), \\ \dot y &= A_{Q^1} y + (I - \pi) X_0 \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_2(\Phi_1(-\tau)z + y_1)^2 + a_3(\Phi_1(-\tau)z + y_1)^3) \end{bmatrix}. \end{aligned} \tag{41}$$

4.2. Unfolding of the nth-order equation

We choose the following unfolding for the nth-order delay-differential equation:

$$L(v_1, v_2)(u_t, v_1, \dots, v_{n-1}) = \begin{bmatrix} v_1 \\ \vdots \\ v_{n-2} \\ -\beta_1 v_{n-1} - \cdots - (\beta_1 - v_1)v_1 + (a_1 + v_2)u(t-\tau) \end{bmatrix}.$$

Thus,

$$(L(v_1, v_2) - L_0)\Phi z = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ iv_1(\omega_1(z_1 - \bar{z}_1) + \omega_2(z_2 - \bar{z}_2))) + v_2\Phi_1(-\tau)z \end{bmatrix}.$$

The quadratic terms computed from (5) are given by

$$\Psi(0)[L(v_1, v_2) - L_0]\Phi z = \Psi_n(0)(iv_1(\omega_1(z_1 - \bar{z}_1) + \omega_2(z_2 - \bar{z}_2)) + v_2\Phi_1(-\tau)z).$$

The normal form to degree two, given by Eq. (12) and Proposition 2.1, is

$$\begin{aligned} \dot{z}_1 &= i\omega_1 z_1 + (i\Psi_n^1(0)\omega_1)v_1 z_1 + (\Psi_n^1(0)e^{i\omega_1\tau})v_2 z_1, \\ \dot{z}_2 &= i\omega_2 z_2 + (i\Psi_n^3(0)\omega_2)v_1 z_2 + (\Psi_n^3(0)e^{i\omega_2\tau})v_2 z_2, \end{aligned}$$

and after transformation to polar coordinates the radial part becomes

$$\begin{aligned} \dot{r}_1 &= (\omega_1 \mathrm{Re}(i\Psi_n^1(0))v_1 + \mathrm{Re}(\Psi_n^1(0)e^{i\omega_1\tau})v_2)r_1, \\ \dot{r}_2 &= (\omega_2 \mathrm{Re}(i\Psi_n^3(0))v_1 + \mathrm{Re}(\Psi_n^3(0)e^{i\omega_2\tau})v_2)r_2. \end{aligned}$$

Lemma 4.2. *Generically, the independent unfolding parameters* (v_1, v_2) *of* (38) *map to independent unfolding parameters* (μ_1, μ_2) *of the normal form equations.*

Proof. As in Proposition 3.2, it is easy to check that if $\omega_1 \neq \omega_2$, the determinant of

$$\begin{bmatrix} \omega_1 \mathrm{Re}(i\Psi_n^1(0)) & \mathrm{Re}(\Psi_n^1(0)e^{i\omega_1\tau}) \\ \omega_2 \mathrm{Re}(i\Psi_n^3(0)) & \mathrm{Re}(\Psi_n^3(0)e^{i\omega_2\tau}) \end{bmatrix}$$

is nonzero for an open and dense set of values of τ. $\square$

In particular, note that it is necessary to have a parameter as coefficient of the $u(t-\tau)$ term while the other unfolding parameter can be chosen in front of any other term.

4.3. Normal form of the nth-order scalar equation

In this section, we discuss the normal form of the nth-order scalar delay-differential equation (36). We proceed with normal form transformations of (41). Consider the normal form transformation for quadratic terms

$$(z, y) = (\tilde{z}, \tilde{y}) + (S_2(\tilde{z}), T_2(\tilde{z})), \tag{42}$$

where $T_2(z) = [T_2^1(z), \ldots, T_2^n(z)]^t$. After this transformation the $\dot{z}$ equation becomes

$$\dot{z} = Bz + \Psi_n(0)\left\{a_3(\Phi_1(-\tau)z)^3 + 2a_2\left[\frac{\partial \Phi_1(-\tau)z}{\partial z_1} S_2^1(z) + \frac{\partial \Phi_1(-\tau)z}{\partial \bar{z}_1} S_2^2(z)\right.\right.$$

$$\left.\left. + \frac{\partial \Phi_1(-\tau)z}{\partial z_2} S_2^3(z) + \frac{\partial \Phi_1(-\tau)z}{\partial \bar{z}_2} S_2^4(z)\right] + 2a_2\Phi_1(-\tau)(z)T_2^1(z)\right\}. \tag{43}$$

This equation is very similar to the $\dot{z}$ equation of the scalar first-order equation (16) in normal form to cubic order. Hence, modulo the computation of $T_2^1(z)$, the cubic coefficients c_{ij} are given by Proposition 3.7.

We now prove (43). The cubic terms after normal form transformation (42) are given by

$$\tilde{F}_3(z) = F_3(\Phi z) + (d_z F_2(\Phi_1(-\tau)z + y_1))|_{y_1=0} S_2(z) + (d_y F_2(\Phi_1(-\tau)z + y_1))|_{y_1=0} T_2(z).$$

Now,

$$F_3(\Phi_1(-\tau)z) + (d_y F_2(\Phi_1(-\tau)z + y_1))|_{y_1=0} T_2(z)$$

$$= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_3(\Phi_1(-\tau)z)^3 + 2a_2\Phi_1(-\tau)(z)T_2^1(z) \end{bmatrix},$$

$$d_z F_2(\Phi_1(-\tau)z) S_2(z) = 2a_2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ \dfrac{\partial \Phi_1(-\tau)z}{\partial z_1} & \dfrac{\partial \Phi_1(-\tau)z}{\partial \bar{z}_1} & \dfrac{\partial \Phi_1(-\tau)z}{\partial z_2} & \dfrac{\partial \Phi_1(-\tau)z}{\partial \bar{z}_2} \end{bmatrix} \begin{bmatrix} S_2^1 \\ S_2^2 \\ S_2^3 \\ S_2^4 \end{bmatrix}.$$

Thus $\dot{z}$ is given by (43) where only $T_2^1(z)$ enters in the cubic terms after normal form transformation of the quadratic terms of the dy/dt equation.

4.3.1. Computation of T_2

In the case of the nth-order Eq. (36), Eq. (32) for the quadratic terms is

$$D_zT_2(z)Bz - A_{Q^1}(T_2(z)) = (I-\pi)X_0\begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_2(\Phi_1(-\tau)z)^2 \end{bmatrix} = \tilde{f}_2(z). \tag{44}$$

Recall that $A_{Q^1}y = \dot{y} + X_0(L(y) - \dot{y}(0))$ and $(I-\pi)X_0 = X_0 - \Phi\Psi(0)$. Thus, (44) reduces to solving for $T_2(z)$ the system

$$D_zT_2(z)Bz - \dot{T}_2(z) = -\Phi\Psi(0)\tilde{f}_2(z) \tag{45}$$

with boundary conditions

$$-\dot{T}_2(z)(0) + L(T_2(z)) = \tilde{f}_2(z). \tag{46}$$

The n components of (45) are given by

$$-\omega_1\left[\frac{\partial T_2^j}{\partial z_1}\bar{z}_1 - \frac{\partial T_2^j}{\partial \bar{z}_1}z_1\right] - \omega_2\left[\frac{\partial T_2^j}{\partial z_2}\bar{z}_2 - \frac{\partial T_2^1}{\partial \bar{z}_2}z_2\right] - \dot{T}_2^1(z) = -\Phi_j\Psi_n(0)a_2\Phi_1(z)^2, \tag{47}$$

where j runs from 1 to n. In particular,

$$-\omega_1\left[\frac{\partial T_2^1}{\partial z_1}\bar{z}_1 - \frac{\partial T_2^1}{\partial \bar{z}_1}z_1\right] - \omega_2\left[\frac{\partial T_2^1}{\partial z_2}\bar{z}_2 - \frac{\partial T_2^1}{\partial \bar{z}_2}z_2\right] - \dot{T}_2^1(z) = -\Phi_1\Psi_n(0)a_2\Phi_1(z)^2.$$

The only difference lies in solving the boundary conditions for $T_2^1(z)$ which involves the knowledge of $T_2^j(z)$ for $j = 2, \dots, n$. We know already that the c_{ij} coefficients in the case of the nth-order equation are identical to the c_{ij} coefficients of Proposition 3.7 up to the T_2^1 term. Consider now the boundary conditions:

$$\begin{cases} \dot{T}_2^1(z)(0) - T_2^2(z)(0) = 0, \\ \quad \vdots \\ \dot{T}_2^{n-1}(z)(0) - T_2^n(z)(0) = 0, \\ \dot{T}_2^n(z)(0) + \beta_1 T_2^{n-1}(z)(0) + \cdots + \beta_n T_2^1(z)(0) - a_1 T_2^1(z)(-\tau) = a_2(\Phi_1(-\tau)z)^2. \end{cases} \tag{48}$$

Since (36) has constant coefficients, Eq. (48) factors into subsystems

$$\begin{cases} \dot{h}^1_{(q1,q2,q3,q4)}(0) - h^2_{(q1,q2,q3,q4)} = 0, \\ \vdots \\ \dot{h}^{n-1}_{(q1,q2,q3,q4)}(0) - h^n_{(q1,q2,q3,q4)} = 0, \\ \dot{h}^n_{(q1,q2,q3,q4)}(0) + \cdots + \beta_n h^1_{(q1,q2,q3,q4)}(0) - a_1 h_{(q1,q2,q3,q4)}(-\tau) = a_2 \xi_{(q1,q2,q3,q4)}, \end{cases} \tag{49}$$

where $\xi_{(q1,q2,q3,q4)}$ is the coefficient of z with power $(q1, q2, q3, q4)$ in $(\Phi_1(-\tau)z)^2$.

Proposition 4.3. *The polynomial* $T^1_2(z)$ *found by solving* (47) *and* (49) *is of the same form as* σ^2_2 *in Proposition* 3.6.

Proof. See Lemmas A.1 and A.6 in the appendix. □

Therefore the following result follows.

Proposition 4.4. *The coefficients of the cubic terms of the normal form of* (36) *are given by Proposition* 3.7 *where the polynomials* P_1, P_2, Q_1 *and* Q_2 *depend on the boundary condition* (48).

Proof. The proof follows from Eq. (43) and Proposition 4.3. □

Proposition 4.4 implies that Proposition 3.9 applies directly to *n*th-order scalar delay equations.

5. Discussion

We have presented an analysis of the relationship between projected flows associated with ordinary differential equations on centre manifolds and the delay-differential equation from which they originate, in the case of a double Hopf bifurcation. We have seen that the universal unfolding of the vector field around the singular point may or may not have restrictions, moreover restrictions are also influenced by the modelling context in which the delay equation arises. As pointed out in [13], there is a difference between unfolding such a singularity in general, and unfolding in the context of modelling using a specific class of delay-differential equations.

Indeed, the restrictions introduced by the specific structure of the model put conditions on the possible range of parameters allowed in the unfolding. The ensuing range of invariant sets is thus limited by the framework in which the model is developed. This shifts some of the burden of the analysis from the purely mathematical considerations to the derivation of the model itself. It thus becomes paramount to have a properly derived system of functional differential equations to adequately translate the biological or mechanical system under study.

Our analysis is the first one addressing the double Hopf bifurcations. Previous investigations [6,8,9,16] have considered simpler bifurcations, all leading to centre manifolds of dimension three or less. We have made use of symmetric bifurcation techniques, explaining in general terms the intriguing simplifying relation, discovered in [2], relating the two cubic terms in the scalar first-order equation with two delays. The role of the symmetry of the hyperbolic tangent in that analysis becomes transparent with the calculations presented here.

We have only studied, albeit in some details, scalar equations of arbitrary order. The only caveat is the necessity for a double Hopf bifurcation point to exist, which is impossible in the case of a first-order equation with a single delay. The same formal analysis can be extended to *systems* of functional differential equations. Our preliminary calculations indicate a fundamental increase in algebraic difficulties, not all of which can be overcome by the use of symbolic manipulation software, such as MAPLE. It is hard to predict how much of our analysis can thus be extended to large-scale systems.

What is clear, though, is the benefit from this investigation for the purposes of modelling biological systems using delay differential equations, and the insight provided into the possible behaviours around singular stationary solutions of the delay equations.

Acknowledgments

The research presented here was supported by the Natural Sciences and Engineering Research Council (NSERC, Canada) (Postdoctoral Fellowship to PLB, Research Grant to JB), the Fonds pour la Formation de Chercheurs et l'Aide à la recherche (FCAR, Québec) (Team Grant to JB) and the Comité d'étude et d'administration de la recherche (CEDAR, Université de Montréal) (Centre Grant to the CRM).

Appendix. Proof of Propositions 3.6 and 4.3

To prove Propositions 3.6 and 4.3 we need to solve equations for σ_2^2 and T_2. We begin by writing Eqs. (33) and (48) in a suitable form for easy integrating. The integration is done in the lemmae that follow and the boundary conditions are used to determine the integrating constants. We write the defining condition equations for σ_2^2 and T_j^2 for all j:

$$-\omega_1\left[\frac{\partial\sigma_2^2}{\partial z_1}\bar{z}_1-\frac{\partial\sigma_2^2}{\partial\bar{z}_1}z_1\right]-\omega_2\left[\frac{\partial\sigma_2^2}{\partial z_2}\bar{z}_2-\frac{\partial\sigma_2^2}{\partial\bar{z}_2}z_2\right]-\dot{\sigma}_2^2(x)+X_0[\dot{\sigma}_2^2(0)-L(\sigma_2^2(x))] \tag{A.1}$$

$$=a_2[X_0-\Phi\Psi(0)](\Phi_1(-\tau)z)^2, \tag{A.2}$$

where Φ stands for Φ in (33) and it stands for Φ_j in (48). Similarly $\Psi(0)$ stands for $\Psi(0)$ in (33) and for $\Psi_n(0)$ in (48). Recall that $\Phi_1 = (e^{i\omega_1\theta}, e^{-i\omega_1\theta}, e^{i\omega_2\theta}, e^{-i\omega_2\theta})$.

Eq. (A.1) is split into two linear differential equations:

$$-\omega_1\left[\frac{\partial\sigma_2^2}{\partial z_1}\bar{z}_1 - \frac{\partial\sigma_2^2}{\partial\bar{z}_1}z_1\right] - \omega_2\left[\frac{\partial\sigma_2^2}{\partial z_2}\bar{z}_2 - \frac{\partial\sigma_2^2}{\partial\bar{z}_2}z_2\right] - \dot{\sigma}_2^2(x) = -a_2\Phi\Psi(0)(\Phi_1(-\tau)z)^2 \tag{A.3}$$

and

$$\dot{\sigma}_2^2(0) - L(\sigma_2^2(x)) = a_2(\Phi(-\tau)z)^2.$$

Let $\dot{h}$ be differentiation with respect to θ. Eq. (A.3) can be written in matrix form

$$-\dot{h} = Ah + f, \tag{A.4}$$

where $f = -a_2\Phi\Psi(0)(\Phi_1(-\tau)z)^2$,

$$h = (h_{2,0,0,0}, h_{0,2,0,0}, h_{0,0,2,0}, h_{0,0,0,2}, h_{1,1,0,0}, h_{0,0,1,1}, h_{1,0,1,0}, h_{0,1,0,1}, h_{0,1,1,0}, h_{1,0,0,1})$$

and

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & \omega_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\omega_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\omega_2 & 0 & 0 & 0 & 0 \\ -2\omega_1 & 2\omega_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\omega_2 & 2\omega_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_1 & \omega_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\omega_2 & -\omega_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\omega_1 & \omega_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\omega_2 & \omega_1 & 0 & 0 \end{bmatrix}.$$

Now

$$\Phi\Psi(0) = \mathrm{Re}(\psi_1(0)e^{i\omega_1\theta}) - \mathrm{Re}(\psi_3(0)e^{i\omega_2\theta}),$$

and set $\mathrm{Re}(\psi_1(0)) = \xi(\omega_1)$, $\mathrm{Re}(\psi_3(0)) = \xi(\omega_2)$, $\mathrm{Im}(\psi_1(0)) = \zeta(\omega_1)$ and $\mathrm{Im}(\psi_3(0)) = \zeta(\omega_2)$ for some ξ and ζ. Let $\mathscr{H}(a) = \xi(a)\cos(a\theta) - \zeta(a)\sin(a\theta)$, then $\Phi\Psi(0) = \mathscr{H}(\omega_1) + \mathscr{H}(\omega_2)$. Since $\xi(a) = \xi(-a)$ and $\zeta(-a) = -\zeta(a)$ then $\mathscr{H}$ is an even function and so is $\Phi\Psi(0)$. Since $\mathscr{H}(\theta, -\omega_1) = \mathscr{H}(\theta, \omega_1)$, then

$$-\dot{h}_{2,0,0,0} = \omega_1 h_{1,1,0,0} - (\mathscr{H}(\theta,\omega_1) + \mathscr{H}(\theta,\omega_2))a_2e^{2i\omega_1\tau},$$
$$-\dot{h}_{0,2,0,0} = (-\omega_1)h_{1,1,0,0} - (\mathscr{H}(\theta,-\omega_1) + \mathscr{H}(\theta,\omega_2))a_2e^{-2i\omega_1\tau}.$$

Therefore, $h_{0,2,0,0}(\theta,\omega_1)=h_{2,0,0,0}(\theta,-\omega_1)$. The same relationship holds between $h_{0,0,2,0}$ and $h_{0,0,0,2}$ but with ω_1 replaced by ω_2. The system then reduces to two four-dimensional systems.

$$-\dot{h}_1=A_1h_1+f_1 \quad \text{and} \quad -\dot{h}_2=A_2h_2+f_2, \tag{A.5}$$

where

$$\begin{aligned}
&h_1=(h_{2,0,0,0},h_{0,0,2,0},h_{1,1,0,0},h_{0,0,1,1}), \quad h_2=(h_{1,0,1,0},h_{0,1,0,1},h_{0,1,1,0},h_{1,0,0,1}),\\
&f_1=-a_2\Phi\Psi(0)(e^{2i\omega_1\tau},e^{2i\tau\omega_2},2,2)^t,\\
&f_2=-a_2\Phi\Psi(0)(2e^{i\tau(\omega_1+\omega_2)},2e^{-i\tau(\omega_1+\omega_2)},2e^{-i\tau(\omega_1-\omega_2)},2e^{i\tau(\omega_1-\omega_2)})^t
\end{aligned}$$

and the matrices are

$$A_1=\begin{bmatrix} 0 & 0 & \omega_1 & 0\\ 0 & 0 & 0 & \omega_2\\ -2\omega_1 & 0 & 0 & 0\\ 0 & -2\omega_2 & 0 & 0\end{bmatrix}, \qquad A_2=\begin{bmatrix} 0 & 0 & \omega_1 & \omega_2\\ 0 & 0 & -\omega_2 & -\omega_1\\ -\omega_1 & \omega_2 & 0 & 0\\ -\omega_2 & \omega_1 & 0 & 0\end{bmatrix}.$$

The boundary conditions are

$$\begin{aligned}
&\dot{h}_1(0)-L(h_1)=(a_2e^{2i\omega_1\tau},a_2e^{2i\tau\omega_2},2a_2,2a_2)^t,\\
&\dot{h}_2(0)-L(h_2)=(2a_2e^{i\tau(\omega_1+\omega_2)},2a_2e^{-i\tau(\omega_1+\omega_2)},2a_2e^{-i\tau(\omega_1-\omega_2)},2a_2e^{i\tau(\omega_1-\omega_2)})^t. \qquad (A.6)
\end{aligned}$$

The following lemma gives h.

Lemma A.1. *The solutions to Eqs.* (A.5) *are*

$$-h_1(\theta)=e^{\theta A_1}K_1+e^{\theta A_1}\left(\int_0^\theta e^{-sA_1}f_1\right), \qquad -h_2(\theta)=e^{\theta A_2}K_2+e^{\theta A_2}\left(\int_0^\theta e^{-sA_2}f_2\right),$$

where

$$\begin{aligned}
&K_1=a_2(\tilde{\mathscr{A}}(\omega_1),\tilde{\mathscr{A}}X(\omega_2),\tilde{\mathscr{B}}(\omega_1),\tilde{\mathscr{B}}(\omega_2))^t,\\
&K_2=a_2(\tilde{\chi}_1(\omega_1,\omega_2),\tilde{\chi}_1(-\omega_1,-\omega_2),\tilde{\chi}_2(\omega_1,\omega_2),\tilde{\chi}_2(-\omega_1,-\omega_2))^t,\\
&\textstyle\int_0^\theta e^{-sA_1}f_1\,ds=a_2(\mathscr{A}(\theta,\omega_1,\omega_2),\mathscr{A}(\theta,\omega_2,\omega_1),\mathscr{B}(\theta,\omega_1,\omega_2),\mathscr{B}(\theta,\omega_2,\omega_1))^t,
\end{aligned}$$

$$\int_0^\theta e^{-sA_2}f_2\,ds=a_2(\chi_1(\theta,\omega_1,\omega_2),\chi_1(\theta,-\omega_1,-\omega_2),\chi_2(\theta,\omega_1,\omega_2),\chi_2(\theta,-\omega_1,-\omega_2))^t,$$

$$\exp(\theta A_1)=\begin{bmatrix} \cos(\sqrt{2}\omega_1\theta) & 0 & \frac{1}{2}\sqrt{2}\sin(\sqrt{2}\omega_1\theta) & 0\\ 0 & \cos(\sqrt{2}\omega_2\theta) & 0 & \frac{1}{2}\sqrt{2}\sin(\sqrt{2}\omega_2\theta)\\ -\sqrt{2}\sin(\sqrt{2}\omega_1\theta) & 0 & \cos(\sqrt{2}\omega_1\theta) & 0\\ 0 & -\sqrt{2}\sin(\sqrt{2}\omega_2\theta) & 0 & \cos(\sqrt{2}\omega_2\theta)\end{bmatrix}$$

and

$$\exp(\theta A_2)$$
$$= \begin{bmatrix} \cos(\omega_1\theta)\cos(\omega_2\theta) & \sin(\omega_1\theta)\sin(\omega_2\theta) & \sin(\omega_1\theta)\cos(\omega_2\theta) & \cos(\omega_1\theta)\sin(\omega_2\theta) \\ \sin(\omega_1\theta)\sin(\omega_2\theta) & \cos(\omega_1\theta)\cos(\omega_2\theta) & -\cos(\omega_1\theta)\sin(\omega_2\theta) & -\sin(\omega_1\theta)\cos(\omega_2\theta) \\ -\sin(\omega_1\theta)\cos(\omega_2\theta) & \cos(\omega_1\theta)\sin(\omega_2\theta) & \cos(\omega_1\theta)\cos(\omega_2\theta) & -\sin(\omega_1\theta)\sin(\omega_2\theta) \\ -\cos(\omega_1\theta)\sin(\omega_2\theta) & \sin(\omega_1\theta)\cos(\omega_2\theta) & -\sin(\omega_1\theta)\sin(\omega_2\theta) & \cos(\omega_1\theta)\cos(\omega_2\theta) \end{bmatrix}.$$

Proof. The proof follows from the following Lemmas A.2, A.3 and A.5. $\square$

Lemma A.2.

$$\int_0^\theta e^{-sA_1} f_1\, ds = a_2(\mathscr{A}(\theta,\omega_1,\omega_2), \mathscr{A}(\theta,\omega_2,\omega_1), \mathscr{B}(\theta,\omega_1,\omega_2), \mathscr{B}(\theta,\omega_2,\omega_1))^t,$$
$$\int_0^\theta e^{-sA_2} f_2\, ds = a_2(\chi_1(\theta,\omega_1,\omega_2), \chi_1(\theta,-\omega_1,-\omega_2), \chi_2(\theta,\omega_1,\omega_2), \chi_2(\theta,-\omega_1,-\omega_2))^t.$$

Proof. We consider first the integral

$$\int_0^\theta e^{-sA_1} f_1\, ds = a_2 \int_0^\theta e^{-sA_1}(-\Phi\Psi(0))(e^{2i\omega_1\tau}, e^{2i\omega_2\tau}, 2, 2)^t ds.$$

which separates into four integrals where $-\Phi\Psi(0) = \mathscr{H}(s,\omega_1) + \mathscr{H}(s,\omega_2)$. Each integral is of the form

$$\int_0^\theta (\mathscr{H}(s,\omega_1) + \mathscr{H}(s,\omega_2))\mathscr{J}(s,a)\, ds,$$

for some function $\mathscr{J}$ where $a = \omega_1$ or $a = \omega_2$. Hence it is easy to see that if

$$\mathscr{I}(\omega_1,\omega_2) = \int_0^\theta (\mathscr{H}(s,\omega_1) + \mathscr{H}(s,\omega_2))\mathscr{J}(s,\omega_1)\, ds$$

then

$$\mathscr{I}(\omega_2,\omega_1) = \int_0^\theta (\mathscr{H}(s,\omega_1) + \mathscr{H}(s,\omega_2))\mathscr{J}(s,\omega_2)\, ds.$$

It is easy to check that

$$e^{-sA_2}(2e^{i\tau(\omega_1+\omega_2)}, 2e^{-i\tau(\omega_1+\omega_2)}, 2e^{-i\tau(\omega_1-\omega_2)}, 2e^{i\tau(\omega_1-\omega_2)})^t$$
$$= (\delta_1(s,\omega_1,\omega_2), \delta_1(s,-\omega_1,-\omega_2), \delta_2(s,\omega_1,\omega_2), \delta_2(s,-\omega_1,-\omega_2))^t,$$

thus

$$\int_0^\theta e^{-sA_2} f_2\, ds = \int_0^\theta (\mathscr{H}(s,\omega_1) + \mathscr{H}(s,\omega_2))(\delta_1(s,\omega_1,\omega_2), \\ \times \delta_1(s,-\omega_1,-\omega_2), \delta_2(s,\omega_1,\omega_2), \delta_2(s,-\omega_1,-\omega_2))^t\, ds$$

Since $\mathscr{H}(s,a)$ is even in a, then

$$\mathscr{H}(\omega_1,\omega_2)\delta_i(s,-\omega_1,-\omega_2) = \mathscr{H}(-\omega_1,-\omega_2)\delta_i(s,-\omega_1,-\omega_2) = \mathscr{H}(a,b)\delta_i(s,a,b),$$

for $i = 1, 2$, where $a = -\omega_1$ and $b = -\omega_2$ and the result follows. □

Lemma A.3. *The sets of matrices $\mathscr{A}_1$ and $\mathscr{A}_2$, respectively, of the form*

$$M_1 = \begin{bmatrix} a & 0 & b & 0 \\ 0 & c & 0 & d \\ -2b & 0 & a & 0 \\ 0 & -2d & 0 & c \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} x & y & z & w \\ y & x & -w & -z \\ -z & w & x & -y \\ -w & z & -y & x \end{bmatrix},$$

where $x \neq \pm y$ and $z \neq \pm w$ for all nonzero matrices, are fields. Moreover,

$$M_1(\alpha(\omega_1,\omega_2), \alpha(\omega_2,\omega_1), \beta(\omega_1,\omega_2), \beta(\omega_2,\omega_1))^t \\ = (\alpha_1(\omega_1,\omega_2), \alpha_1(\omega_2,\omega_1), \beta_1(\omega_1,\omega_2), \beta_1(\omega_2,\omega_1))^t. \tag{A.7}$$

Proof. The determinants are $\det(M_1) = (a^2 + 2b^2)(c^2 + 2d^2)$ and $\det(M_2) = ((z - w)^2 + (x+y)^2)((y-x)^2 + (z+w)^2)$ which vanish only for the zero matrix. Commutativity and property (A.7) are verified by a simple computation. □

Remark A.4. Note that in Lemma A.1, $\exp(\theta A_1)$ is an element of $\mathscr{A}_1$ and $\exp(\theta A_2)$ is in $\mathscr{A}_2$ since $A_1 \in \mathscr{A}_1$ and $A_2 \in \mathscr{A}_2$. Moreover, if $M_2 = A_2^m$ for any integer $m \geqslant 0$ or $M_2 = \exp(\theta A_2)$, then an easy calculation shows that

$$M_2(\chi(\omega_1,\omega_2), \chi(-\omega_1,-\omega_2), \xi(\omega_1,\omega_2), \xi(-\omega_1,-\omega_2))^t \\ = (\chi_1(\omega_1,\omega_2), \chi_1(-\omega_1,-\omega_2), \xi_1(\omega_1,\omega_2), \xi_1(-\omega_1,-\omega_2))^t. \tag{A.8}$$

Proof of Proposition 3.6. From Lemmas A.1 and A.3 we see that the multiplication and addition in the expressions for h yield the desired result. □

Lemma A.5. *The constants K_1 and K_2 found using the boundary conditions with $L(0,0)$ coming from* (20) *have the form*

$$\begin{aligned} K_1 &= a_2(\tilde{\mathscr{A}}(\omega_1), \tilde{\mathscr{A}}(\omega_2), \tilde{\mathscr{B}}(\omega_1), \tilde{\mathscr{B}}(\omega_2)), \\ K_2 &= a_2(\tilde{\chi}_1(\omega_1,\omega_2), \tilde{\chi}_1(-\omega_1,-\omega_2), \tilde{\chi}_2(\omega_1,\omega_2), \tilde{\chi}_2(-\omega_1,-\omega_2)). \end{aligned}$$

Proof. Writing the boundary equation using the solutions $h_1(\theta)$ computed before we obtain

$$\begin{aligned} &(A_1 - a_{10}e^{-\tau_1 A_1} - a_{01}e^{-\tau_2 A_1})K_1 \\ &\quad = f_1(0) + a_{10}e^{-\tau_1 A_1}\int_0^{-\tau_1} e^{-sA_1} f_1(s)\,ds + a_{01}e^{-\tau_2 A_1}\int_0^{-\tau_2} e^{-sA_1} f_1(s)\,ds \\ &\qquad + (e^{2i\omega_1\tau}, e^{2i\omega_2\tau}, 2, 2)^t \\ &\quad = (\alpha(\omega_1,\omega_2), \alpha(\omega_2,\omega_1), \beta(\omega_1,\omega_2), \beta(\omega_2,\omega_1))^t, \end{aligned}$$

where the last equality is easily shown using Lemma A.2. By Lemma A.3 $(A_1 - a_{10}e^{-\tau_1 A_1} - a_{01}e^{-\tau_2 A_1})^{-1}$ is of the form M_1 and the result follows. The vector K_2 is computed in the same way using Remark A.4. $\square$

Lemma A.6. *The constants K_1 and K_2 found using the boundary conditions* (49) *have the same form as in Lemma* A.5.

Proof. Let

$$T_2^j(z) = \sum_{|q|=2} h^j_{q_1,q_2,q_3,q_4}(\theta) z_1^{q_1}\,\bar{z}_1^{q_2}\,z_2^{q_3}\,\bar{z}_2^{q_4},$$

where $j = 1, \ldots, n$, $|q| = q_1 + q_2 + q_3 + q_4$ and $h^j_{q_1,q_2,q_3,q_4}(\theta) \in Q^1$. For $j = 1, \ldots, n$, let

$$h_1^j = (h^j_{2,0,0,0}, h^j_{0,0,2,0}, h^j_{1,1,0,0}, h^j_{0,0,1,1}) \quad \text{and} \quad h_2^j = (h^j_{1,0,1,0}, h^j_{0,1,0,1}, h^j_{0,1,1,0}, h^j_{1,0,0,1}).$$

Then using the solutions of Eq. (A.5) for h_i^j, we replace in the boundary conditions (49). By Lemma A.1, $h_1^j = e^{\theta A_1}(K_1^j + f_1^j(0))$ where the superscripts of K and f are indices setting $K_1^0 = K_1$ and $f_1^j(0)$ has the same form as in Lemma A.2. Thus for h_1^j we obtain

$$\begin{cases} A_1K_1 + f_1(0) + K_1^1 = 0, \\ \quad\vdots \\ A_1K_1^{n-2} + f_1^{n-1}(0) + K_1^{n-1} = 0, \\ A_1K_1^{n-1} + f_1^{n-1}(0) + \beta_1 K_1^{n-1} + \beta_n K_1 + a_1 e^{-\tau A_1} K_1 = a_2(e^{2i\omega_1\tau}, e^{2i\omega_2\tau}, 2, 2)^t). \end{cases}$$

Solve K_1^j in terms of $K_1 : K_1^j = (-1)^j A_1^j K_1 + (-1)^j (A_1^{j-1} f_1(0) - A_1^{j-2} f_1^1(0) + \cdots \pm f_1^{j-1}(0))$. Replacing in the last equation and putting on the right-hand side all the terms which do not contain K_1 we obtain

$$(-1)^n (A_1^n + \beta_1 A_1^{n-1} + \cdots \beta_n I + a_1 e^{-\tau A_1}) K_1$$

$$= a_2 (g_1(\omega_1, \omega_2), g_1(\omega_2, \omega_1), g_2(\omega_1, \omega_2), g_2(\omega_2, \omega_1))^t.$$

Using Lemma A.3 in the preceding equation yields the result. The other vectors h_i^j are handled in the same way and yield similar results. □

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